Chapter 7

THE LOGARITHMIC FUNCTION AND LOGARITHMS

Did you know that the energy in the sound of a jet aircraft engine is about one trillion times greater than the energy you exert when you whisper to your friend seated next to you on the plane that you're secretly afraid of flying? Rather than dealing with such a wide range of values, people who work with sound — whether they be broadcasters, people in the recording industry, or engineers trying to reduce engine noise inside an airplane — all measure sound levels using the more manageable decibel scale, which is an example of a logarithmic scale. Other examples of logarithmic scales include the Richter scale for measuring the intensity of earthquakes and the pH scale for measuring the acid content in a substance. These scales are used to simplify certain phenomena that might vary by large magnitudes, and the scales are all based on logarithmic functions like those that are studied in this chapter.

CHAPTER EXPECTATIONS In this chapter, you will

- define logarithmic function $\log_a x$ ($a > 1$), **Section 7.1**
- express logarithmic equations in exponential form, **Section 7.1**
- simplify and evaluate expressions containing logarithms, **Section 7.2**
- solve exponential and logarithmic equations, **Section 7.3, 7.5**
- solve simple problems involving logarithmic scales, **Section 7.4, Career Link**

Review of Prerequisite Skills

In the last chapter, we examined the exponential function and its use in solving equations. In this chapter, we will study the inverse of the exponential function. This function is the **logarithmic function**.

To begin, we will review the important facts associated with the exponential function:

- For $y = b^x$, $b > 1$, the function is increasing, the y-intercept is 1, and the x-axis is a horizontal asymptote.
- For $y = b^x$, $0 < b < 1$, the function is decreasing, the y-intercept is 1, and the x-axis is a horizontal asymptote.
- If a population is growing at the rate of $i\%$ per year, the function that expresses the population after x years is $f(x) = P_0(1 + i)^x$, where P_0 is the initial population.
- If the population doubles, the base for the exponential function is 2. The function representing the population after t years is $f(x) = P_0 2^{\frac{t}{d}}$, where P_0 is the initial population and d is the doubling time.

Exercise

technology
APPENDIX P. 427

1. Use your graphing calculator to sketch the graph of $y = 3^x$.
   a. State the domain and range.
   b. How is the slope of the graph of $y = 3^x$ related to the slope of the graph of $y = x^3$?
   c. How is the slope of the graph of $y = \left(\frac{1}{3}\right)^x$ related to the slope of the graph of $y = x^{\frac{1}{3}}$?
   d. Explain how the answers to parts **b** and **c** are related.
2. If f is a function defined by $f(x) = b^x$, where $b > 1$, what can be stated about
   a. the sign of $f(x)$?
   b. the growth behaviour of f?
   c. $f(0)$?

3. If f is a function defined by $f(x) = b^x$, where $0 < b < 1$, what can be stated about
   a. the sign of $f(x)$?
   b. the growth behaviour of f?
   c. $f(0)$?
4. The population of a town is 2400. If the population is predicted to grow at the rate of 6% per year, determine the predicted population in 20 years.
5. A culture initially has 2000 bacteria. If the population doubles every 4 h, determine when the population would be 512 000.
6. The half-life of radium is 1620 years.
   a. If a laboratory initially had 5 g of radium, determine how much they would have in 200 years.
   b. How many years would it take until the laboratory had only 4 g of radium?

Career Link investigate

CHAPTER 7: MEASURING ON A LOGARITHMIC SCALE

We often hear the term "order of magnitude" when people are describing the severity of an earthquake, the acidity of a solution, or the loudness of a sound. **Order of magnitude** is actually a very simple concept. One order of magnitude means 10 times larger, two orders of magnitude mean 100 times larger, and so on. Mathematically, the pattern is 10^0, 10^1, 10^2, 10^3, … 10^n with the order of magnitude as the exponent. This exponent is also known as the **logarithm** of the pattern. On a number line, the intensity of an earthquake could be represented as follows:

Logarithm Notation	0	1	2	3	4	5	6	7	8	9
Exponential Notation	10^0	10^1	10^2	10^3	10^4	10^5	10^6	10^7	10^8	10^9
Standard Notation	1	10	100	1000	10 000	100 000	1 000 000	…		

The logarithmic notation on this number line for measuring earthquake intensity is known as the Richter Scale and indicates that an earthquake of magnitude 5 is 100 times more intense than an earthquake of magnitude 3. Other examples of log scales include the pH scale in chemistry and the bel or decibel scale for the measurement of loudness. In this chapter, you will investigate the properties of the logarithm function and its graph, utilize the rules of logarithms to solve exponential equations, and model exponential and polynomial data using logarithms.

Case Study — Seismologist (Earthquake Geologist)

Seismologists play a critical role in assisting structural engineers to design earthquake-proof buildings. Designing buildings to withstand earthquakes first requires the seismologist to model the behaviour of earthquake shockwaves (e.g., magnitude and frequency). Only then can the engineer determine how the beams and columns must be designed to withstand the tensional, compressive, shearing (tearing), and torsion (twisting) forces the earthquake will cause. While most people would identify Vancouver, B.C., as the city most prone to earthquakes in Canada, there are others that might surprise you, such as Ottawa, Ontario, which experiences earthquakes on a regular basis.

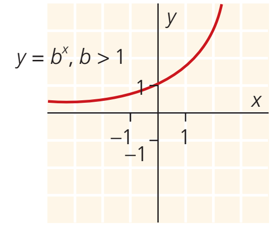
DISCUSSION QUESTIONS

1. Why do you suppose a logarithmic scale is used to model the intensity of earthquakes?
2. How much more intense would an earthquake of magnitude 7.8 be compared to an earthquake of magnitude 4.5? Write down your calculations.
3. Identify at least two other situations when it would be practical to compare events in terms of order of magnitude. Explain your reasoning with specific examples.

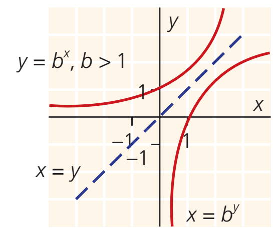
At the end of this chapter, you have an opportunity to demonstrate your learning in exponential and logarithmic functions by completing an analysis of a lake damaged by acid rain. ●

Section 7.1 — The Logarithmic Function

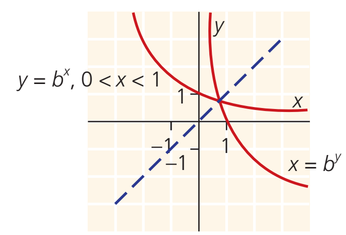
We have been studying the exponential function $f(x) = b^x$, in which $b > 1$. A typical example is shown.



The inverse of the exponential function is obtained by interchanging the x- and y-coordinates.
The inverse of $y = b^x$ *is* $x = b^y$.
The graph of the inverse is obtained by reflection in the line $y = x$.



Recall that if $0 < b < 1$, the graph of $f(x) = b^x$ and its inverse, $x = b^y$, is as shown.



Since the exponential function $y = b^x$ is only defined for $b > 0$, it follows that the inverse function, $x = b^y$, is only defined for $b > 0$. We can also see from the graph that the domain of $x = b^y$ is $x > 0$. We will call this inverse function the **logarithmic function** and write it as $y = \log_b x$. This is read as "y equals log of x to the base b." The function $\log_b x$ is defined only for $x > 0$.

INVESTIGATION The purpose of this investigation is to examine the shape of the graph of $f(x) = \log_b x$. To do so, follow these steps.

Step 1: Consider the function $f(x) = b^x$, where $b = 2$. Prepare a table of values using integer values for the domain $-3 \le x \le 4$.

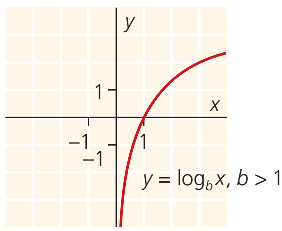
Step 2: Sketch the graph.

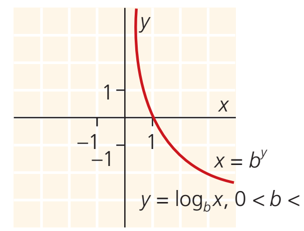
Step 3: Using the image line $y = x$, sketch the graph of the inverse function $f(x) = \log_b x$.

Step 4: By reversing the entries in the table of values from Step 1, compile a table of values for points on the graph $f(x) = \log_b x$ and determine whether or not these points are on the graph of $f(x) = \log_b x$.

Step 5: Repeat Steps 1 to 4 using $b = 3$, $b = \frac{1}{2}$, and $b = \frac{1}{3}$.

As with the exponential function, there are two possible versions of the graph of the logarithmic function:





Properties of the Logarithmic Function $y = \log_b x$

- The base b is positive.
- The x-intercept is 1.
- The y-axis is a vertical asymptote.
- The domain is the set of positive real numbers.
- The range is the set of real numbers.
- The function is increasing if $b > 1$.
- The function is decreasing if $0 < b < 1$.

While any number can be used as a base, the most common base used is 10. Logarithms with base 10 are called **common logarithms**. Common logarithms were used for complicated calculations before the invention of the hand-held calculator in the 1970s. For convenience, $\log_{10} x$ is usually written as just $\log x$, and the base is understood to be 10. Calculators are programmed in base 10, and with this base we can use the calculator to sketch the graph of $y = \log x$.

Exponential Form		**Logarithmic Form**	
$x = b^y$	$\longleftrightarrow$	$y = \log_b x$	$b > 0$ and $b \neq 1$

The logarithm of a number x with a given base is the exponent to which that base must be raised to yield x.

EXAMPLE 1

Change to exponential form.

a. $\log_3 81 = 4$

b. $\log_{25} 5 = \frac{1}{2}$

Solution

a. If $\log_3 81 = 4$, then $3^4 = 81$.

b. If $\log_{25} 5 = \frac{1}{2}$, then $25^{\frac{1}{2}} = 5$.

EXAMPLE 2

Change to logarithmic form.

a. $5^3 = 125$

b. $\left(\frac{1}{2}\right)^{-3} = 8$

Solution

a. If $5^3 = 125$, then $\log_5 125 = 3$.

b. If $\left(\frac{1}{2}\right)^{-3} = 8$, then $\log_{\frac{1}{2}} 8 = -3$.

EXAMPLE 3

Use your calculator to find the value of the following:

a. $\log_{10} 500$

b. $\log 4.6$

c. $\log 0.0231$

Solution

The answers are given to the accuracy of a calculator with a 10-digit display.

a. $\log_{10} 500 = 2.698\,970\,004$

b. $\log 4.6 = 0.662\,757\,831$

c. $\log 0.0231 = -1.636\,388\,02$

EXAMPLE 4

Our first task is to gain an understanding of the arithmetic involving logarithms. Many logarithmic expressions can be evaluated without the use of a calculator. This is particularly useful when the base of the logarithmic function is a number other than 10.

Evaluate the following:

a. $\log_5 25$

b. $\log_3 27$

c. $\log_2\left(\frac{1}{4}\right)$

d. $\log_{\frac{1}{3}} 27$

Solution

a. Let $\log_5 25 = x$.

Then, by definition, $5^x = 25$.

Then $x = 2$ and $\log_5 25 = 2$.

b. Let $\log_3 27 = x$.

Then, by definition, $3^x = 27$.

Then $x = 3$ and $\log_3 27 = 3$.

c. Let $\log_2\left(\frac{1}{4}\right) = x$.

Then $2^x = \frac{1}{4} = \frac{1}{2^2} = 2^{-2}$.

Then $x = -2$ and $\log_2\left(\frac{1}{4}\right) = -2$.

d. Let $\log_{\frac{1}{3}} 27 = x$.

Then $\left(\frac{1}{3}\right)^x = 27$ or $(3^{-1})^x = 3^{-x} = 27$.

Then $x = -3$ and $\log_{\frac{1}{3}} 27 = -3$.

Exercise 7.1

Part A

1. Write each of the following in logarithmic form.

a. $3^2 = 9$ b. $9^0 = 1$ c. $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

d. $36^{\frac{1}{2}} = 6$ e. $27^{\frac{2}{3}} = 9$ f. $2^{-3} = \frac{1}{8}$

Knowledge/Understanding

2. Write each of the following in exponential form.

a. $\log_5 125 = 3$ b. $\log_7 1 = 0$ c. $\log 5\left(\frac{1}{25}\right) = -2$

d. $\log_7\left(\frac{1}{7}\right) = -1$ e. $\log_{\frac{1}{3}} 9 = -2$ f. $\log_9 27 = \frac{3}{2}$

3. Use your calculator to find the value of each of the following:

a. $\log_{10} 37$ b. $\log 0.24$ c. $\log 1000$

d. $\log 52$ e. $\log 1.35$ f. $\log 52648$

Part B

Application

4. On one grid, sketch the graphs of $y = 5^x$ and $y = \log_5 x$.

5. On one grid, sketch the graphs of $y = 5^{-x}$ and $y = \log_{\frac{1}{5}} x$.

Knowledge/Understanding

6. Evaluate each of the following:

a. $\log_2 8$ b. $\log_5 25$ c. $\log_3 81$

d. $\log_7 49$ e. $\log_2\left(\frac{1}{8}\right)$ f. $\log_3\left(\frac{1}{27}\right)$

g. $\log_5 \sqrt{5}$ h. $\log_2 4^2$ i. $\log_2 \sqrt[4]{32}$

Application

7. Evaluate each of the following:

a. $\log_6 36 - \log_5 25$ b. $\log_9\left(\frac{1}{3}\right) + \log_3\left(\frac{1}{9}\right)$ c. $\log_6 \sqrt{36} - \log_{25} 5$

d. $\log_3 \sqrt[4]{27}$ e. $\log_3\left(9 \times \sqrt[5]{9}\right)$ f. $\log_2 16^{\frac{1}{3}}$

Thinking/Inquiry/Problem Solving

8. Use your knowledge of logarithms to solve each of the following equations for x.

a. $\log_5 x = 3$ b. $\log_4 x = 2$ c. $\log_x 27 = 3$

d. $\log_4\left(\frac{1}{64}\right) = x$ e. $\log_x\left(\frac{1}{9}\right) = 2$ f. $\log_{\frac{1}{4}} x = -2$

Communication

9. Explain how you find the value of a logarithm. Give specific examples to illustrate your thought processes.

Part C

Thinking/Inquiry/ Problem Solving

10. Sketch the graph of $y = 3^x + 3^{-x}$, $-4 \leq x \leq 4$. The resulting curve is called a **catenary.**

11. For the function $y = \log_{10}x$, where $0 < x \leq 1000$, how many integer values of y are possible if $y > -20$?

The History of Logarithms

Scottish mathematician John Napier (1550–1617) was the first to define logarithms. Napier, realizing that the base of our number system is 10, used 10 as the base for his logarithms. These logarithms could then be used to make calculations easier.

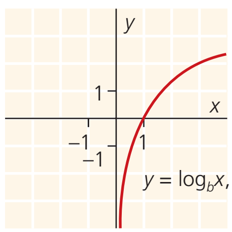
Henry Briggs (1561–1630) saw the practical applications of logarithms to his investigations in trigonometry and astronomy. He spent nine years laboriously calculating a partial table of common logarithms. There was great excitement in the scientific community when this table was published. It simplified the massive calculations of the great astronomers of the day, Tycho Brahe and Johannes Kepler.

Until the mid-1970s, logarithms were used to simplify calculations. Textbooks had tables of common logarithms in the back. Now the hand-held calculator is used instead of logarithms. However, the study of logarithms is still important, because logarithms appear in scientific formulas and in the study of calculus.

Section 7.2 — Properties of Logarithms

In order to simplify and evaluate expressions containing logarithms, we will start to develop an arithmetic of logarithmic expressions by considering some basic ideas.

BASIC PROPERTIES OF LOGARITHMS



INVESTIGATION The purpose of this investigation is to identify basic properties of logarithms.

1. In each of the following, use the definition of a logarithm as an exponent and determine the value of the expression.

a. $\log_5 5$	b. $\log_3 1$	c. $\log_7 7$
d. $\log_4 1$	e. $\log_2 2^5$	f. $\log_3 3^4$
g. $\log 10^{3.6}$	h. $\log 10^{5.78}$	i. $5^{\log_5 25}$
j. $4^{\log_4 64}$	k. $10^{\log 6}$	l. $10^{\log 97}$

2. State a possible value for $\log_b 1$.

3. State a possible value for $\log_b b$.

4. State a possible value for $\log_b b^x$.

5. State a possible value for $b^{\log_b x}$.

Basic Properties of Logarithms

$$\log_b 1 = 0$$
$$\log_b b = 1$$
$$\log_b b^x = x$$
$$b^{\log_b x} = x$$

The proofs of these four basic properties of logarithms are as follows:

1. $\log_b 1 = 0$

   Proof

   Let $\log_b 1 = y$.

   From the definition, $b^y = 1 = b^0$.

   Then $y = 0$ and $\log_b 1 = 0$.

2. $\log_b b = 1$

   Proof

   Let $\log_b b = y$.

   From the definition, $b^y = b = b^1$.

   Then $y = 1$ and $\log_b b = 1$.

3. $\log_b b^x = x$

   Proof

   Let $\log_b b^x = y$.

   From the definition, $b^y = b^x$.

   Then $y = x$ and $\log_b b^x = x$.

4. $b^{\log_b x} = x$

   Proof

   Let $b^{\log_b x} = y$.

   From our knowledge of exponentials, we can write $y = b^t$ for some value of t.

   Then $b^{\log_b x} = b^t$.

   Now $\log_b x = t$.

   From the definition, $b^t = x = y$.

   Then $b^{\log_b x} = x$.

Because logarithms are exponents, the Properties of Logarithms can be derived from the Laws of Exponents. Three properties deal with the logarithm of a product, a quotient, and a power.

1. The logarithm of a product is equal to the sum of the logarithms of the factors. That is,

$$\begin{aligned}
\log_b xw &= \log_b x + \log_b w, \text{ if } x, w > 0.\\
\text{Let } \log_b x &= s \text{ and } \log_b w = t.\\
\text{Then } x &= b^s \text{ and } w = b^t.\\
\text{Therefore, } \log_b xw &= \log_b(b^s \bullet b^t)\\
&= \log_b b^{s+t}\\
&= s + t\\
&= \log_b x + \log_b w
\end{aligned}$$

2. The logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator. That is,

$$\begin{aligned}
\log_b\left(\frac{x}{w}\right) &= \log_b x - \log_b w.\\
\text{Let } \log_b x &= s \text{ and } \log_b w = t.\\
\text{Then } x &= b^s \text{ and } w = b^t.\\
\text{Therefore, } \log_b\left(\frac{x}{w}\right) &= \log_b\left(\frac{b^s}{b^t}\right)\\
&= \log_b b^{s-t}\\
&= s - t\\
&= \log_b x - \log_b w.
\end{aligned}$$

Note that if $x = 1$, $\log_b 1 = 0$ and $\log\left(\frac{1}{w}\right) = -\log_b w$.

3. The logarithm of a number raised to a power is equal to the exponent of the power multiplied by the logarithm of the number. That is,

$$\begin{aligned}
\log_b x^r &= r\log_b x, \text{ when } x > 0 \text{ and } r \text{ is a real number.}\\
\text{Let } \log_b x &= s, \text{ so } x = b^s.\\
\text{Then } \log_b x^r &= \log_b(b^s)^r\\
&= \log_b b^{rs}\\
&= rs\\
&= r\log_b x.
\end{aligned}$$

Note that $\log_b\sqrt[r]{x} = \log_b x^{\frac{1}{r}} = \frac{1}{r}\log_b x$.

Properties of Logarithms

When $x > 0$, $w > 0$, and r is a real number,

$$\log_a xw = \log_a x + \log_a w$$

$$\log_a\left(\frac{x}{w}\right) = \log_a x - \log_a w$$

$$\log_a x^r = r\log_a x.$$

These properties allow us to simplify expressions that might otherwise be complicated.

EXAMPLE 1

Evaluate the following:

a. $\log_{10}(47 \times 512)$ b. $\log_3(81 \times 243)$ c. $\log_4 2 + \log_4 32$

Solution

a. $\log_{10}(47 \times 512) \doteq 4.381\,367\,8$ (by calculator)

b. $\log_3(81 \times 243) = \log_3 81 + \log_3(243) = 4 + 5 = 9$

c. $\log_4 2 + \log_4 32 = \log_4(2 \times 32) = \log_4 64 = 3$

EXAMPLE 2

Simplify the following:

a. $\log_3\left(\frac{27}{81}\right)$ b. $\log_2\left(\frac{75}{26}\right)$ c. $\log_2 48 - \log_2 3$ d. $\log_5 45$

Solution

a. $\log_3\left(\frac{27}{81}\right) = \log_3 27 - \log_3 81 = 3 - 4 = -1,$

or $\log_3\left(\frac{27}{81}\right) = \log_3\left(\frac{1}{3}\right) = \log_3(3^{-1}) = -1$

b. $\log_2\left(\frac{75}{26}\right) = \log_2 75 - \log_2 26$

c. $\log_2 48 - \log_2 3 = \log_2\left(\frac{48}{3}\right) = \log_2 16 = 4$

d. $\log_5 45 = \log_5(5 \times 9) = \log_5 5 + \log_5 9 = 1 + \log_5 9$

EXAMPLE 3

Write $\log_a x^3y^4$ in terms of $\log_a x$ and $\log_a y$.

Solution

$$\begin{aligned}\log_a x^3y^4 &= \log_a x^3 + \log_a y^4\\ &= 3\log_a x + 4\log_a y\end{aligned}$$

EXAMPLE 4

Write $3\log(x + 3) - 2\log(x - 1)$ as a single logarithm.

Solution

$$\begin{aligned}3\log(x + 3) - 2\log(x - 1) &= \log(x + 3)^3 - \log(x - 1)^2\\ &= \log\frac{(x + 3)^3}{(x - 1)^2}\end{aligned}$$

EXAMPLE 5 Evaluate $3^{-\frac{1}{2}\log_3 49}$.

Solution

Consider the exponent first.

$$-\frac{1}{2}\log_3 49 = \log_3(49)^{-\frac{1}{2}}$$
$$= \log_3\left(\tfrac{1}{7}\right)$$

Now, $3^{-\frac{1}{2}\log_3 49} = 3^{\log_3\left(\frac{1}{7}\right)}$

$$= \frac{1}{7}.$$

EXAMPLE 6 Logarithms are particularly useful in solving exponential equations.

Solve each of the following, giving answers to two decimals.

a. $7^x = 400$

b. $7(1.06^x) = 5.20$

Solution

a. $7^x = 400$

Taking logarithms of each side,

$$x \log 7 = \log 400$$
$$x = \frac{\log 400}{\log 7} = 3.08.$$

b. $7(1.06^x) = 5.20$

Taking logarithms of each side,

$$\log 7 + x \log 1.06 = \log 5.20$$
$$x \log 1.06 = \log 5.20 - \log 7$$
$$x = \frac{\log 5.20 - \log 7}{\log 1.06}$$
$$= -5.10.$$

EXAMPLE 7 Describe the relation between the graph of $y = \log_2 x$ and the graph of each of the following:

a. $y = \log_2 x^2$

b. $y = \log_2(4x)$

Solution

a. Since $\log_2 x^2 = 2\log_2 x$, the graph of $y = \log_2 x^2$ is a vertical dilatation of $y = \log_2 x$ by a factor of 2.

b. Since $\log_2 4x = \log_2 4 + \log_2 x = 2 + \log_2 x$, the graph of $y = \log_2 4x$ is a vertical translation of the graph of $y = \log_2 x$ by 2 units upwards.

Exercise 7.2

Part A

1. Write each of the following as a sum of logarithms.

   a. $\log_a(xy)$ b. $\log_m(pq)$

2. Write each of the following as the logarithm of a product.

   a. $\log_a x + \log_a w$ b. $\log_a s + \log_a r$

3. Write each of the following as a difference of logarithms.

   a. $\log_b\left(\frac{x}{y}\right)$ b. $\log_a\left(\frac{r}{s}\right)$

Knowledge/Understanding

4. Write each of the following in the form $m \log_a p$.

   a. $\log_6 13^4$ b. $\log_5 1.3^{-2}$ c. $\log_7 x^{\frac{1}{3}}$ d. $\log_a 6^{-\frac{3}{4}}$

5. Use the properties of logarithms to write each of the following as a sum and/or a difference of logarithms.

   a. $\log_b\left(\frac{xy}{z}\right)$ b. $\log_a\left(\frac{x}{yz}\right)$

Part B

Knowledge/Understanding

6. On the same grid, sketch the graphs of

   a. $y = \log_3 x$ b. $y = \log_3(9x)$ c. $y = \log_3(27x)$ d. $y = \log_3\left(\frac{x}{3}\right)$

7. On the same grid, sketch the graphs of

   a. $y = \log_3 x$ b. $y = \log_3 x^2$ c. $y = \log_3 x^3$ d. $y = \log_3\sqrt{x}$

8. Evaluate each of the following:

   a. $\log_3 135 - \log_3 5$ b. $\log_2 40 + \log_2\left(\frac{4}{5}\right)$ c. $\log_8 640 - \log_8 10$

   d. $\log_5(2.5) + \log_5 10$ e. $\log_2 224 - \log_2 7$ f. $\log_3 36 + \log_3\left(\frac{3}{4}\right)$

Application

9. Evaluate each of the following:

   a. $\log_3 3 + \log_5 1$ b. $\log_3 18 + \log_3\left(\frac{3}{2}\right)$ c. $\log_4 16 - \log_4 1$

   d. $\log_5 5^3$ e. $\log_2 40 - \log_2\left(\frac{5}{2}\right)$ f. $\log_4 4^4 + \log_3 3^3$

   g. $\log_2 14 + \log_2\left(\frac{4}{7}\right)$ h. $\log_5 200 - \log_5 8$ i. $\log_6 4 + \log_6 54$

10. Use the properties of logarithms to write each of the following in terms of $\log_a x$, $\log_a y$, and $\log_a w$.

a. $\log_a \sqrt[3]{x^2y^4}$ b. $\log_a \sqrt{\frac{x^3y^2}{w}}$ c. $\log_a \frac{x^3\,y^4}{\sqrt{x^{\frac{1}{4}}y^{\frac{2}{3}}}}$ d. $\log_a\left(\frac{x^5}{y^3}\right)^{\frac{1}{4}}$

11. Solve each of the following:

a. $10^{2x} = 495$ b. $10^{3x} = 0.473$ c. $10^{-x} = 31.46$

d. $7^x = 35.72$ e. $(0.6)^{4x} = 0.734$ f. $(3.482)^{-x} = 0.0764$

Application

12. Solve each of the following:

a. $12^{2x-3} = 144$ b. $7^{x+9} = 56$ c. $5^{3x+4} = 25$

d. $10^{2x+1} = 95$ e. $6^{x+5} = 71.4$ f. $3^{5-2x} = 875$

13. Solve each of the following. Write your answers correct to two decimal places.

a. $2 \times 3^x = 7 \times 5^x$ b. $12^x = 4 \times 8^{2x}$

c. $4.6 \times 1.06^{2x+3} = 5 \times 3^x$ d. $2.67 \times 7.38^x = 9.365^{x-2}$

e. $12 \times 6^{2x-1} = 11^{x+3}$ f. $7 \times 0.43^{2x} = 9 \times 6^{-x}$

g. $5^x + 3^{2x} = 92$ h. $4 \times 5^x - 3(0.4)^{2x} = 11$

Application

14. Write each of the following as a single logarithm.

a. $\frac{1}{3}\log_a x + \frac{1}{4}\log_a y - \frac{2}{5}\log_a w$ b. $(4\log_5 x - 2\log_5 y) \div 3\log_5 w$

Thinking/Inquiry/ Problem Solving and Communication

15. Describe the transformation that takes the graph of the first function to that of the second.

a. $y = \log x$ and $y = \log(10x)$

b. $y = \log_2 x$ and $y = \log_2(8x^2)$

c. $y = \log_3 x$ and $y = \log_3(27x^3)$

Part C

Thinking/Inquiry/ Problem Solving

16. Evaluate each of the following:

a. $\log_3(27 \bullet \sqrt[3]{81}) + \log_5(125 \bullet \sqrt[4]{5})$ b. $\log_4(2 \bullet \sqrt{32}) + \log_{27}\sqrt{3}$

Communication

17. a. If $y = 3\log x$, what happens to the value of y if

i) x is multiplied by 2?

ii) x is divided by 2?

b. If $y = 5\log x$, what happens to the value of y if

i) x is replaced by $4x$?

ii) x is replaced by $\frac{x}{5}$?

Section 7.3 — Solving Logarithmic Equations

The properties of logarithms we learned in the last section can help us solve equations involving logarithmic expressions. We must remember that $y = \log_a x$ is defined only for $x > 0$. Some of the logarithmic equations we solve will appear to have a root that is less than zero. Such a root is inadmissible. This means that every time we solve a logarithmic equation, we must check that the roots obtained are admissible.

EXAMPLE 1

Solve $\log_6 x = 2$.

Solution

$\log_6 x = 2$

Then $x = 6^2$

$x = 36.$

The root of the equation is $x = 36$.

Checking, $\log_6 36 = 2$, therefore the root is admissible.

EXAMPLE 2

Solve $\log_6 x + \log_6(x + 1) = 1$.

Solution

Simplifying, $\log_6 x + \log_6(x + 1)$

$= \log_6(x(x + 1))$

$= \log_6(x^2 + x).$

Then $\log_6(x^2 + x) = 1$.

In exponential form, $x(x + 1) = 6^1$

$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0$$
$$x = -3 \text{ or } x = 2.$$

The logarithm of a negative number is **not** defined.

Therefore, the root $x = -3$ is inadmissible.

If $x = 2$,

L.S. $= \log_6 x + \log_6(x + 1)$ R.S. $= 1$

$= \log_6(2) + \log_6(3)$

$= \log_6(2 \times 3)$

$= \log_6 6$

$= 1.$

The only root of the equation is $x = 2$.

In the last chapter, we considered some problems involving the exponential function in which it was difficult to solve the resulting equation. Using logarithms makes this process much easier.

EXAMPLE 3 Solve $3^x = 23$.

Solution

Take the logarithm of each side.

$$\log 3^x = \log 23$$

Now use the logarithmic properties to simplify and isolate the variable x.

$$x \log 3 = \log 23$$

$$x = \frac{\log 23}{\log 3}$$

This is the exact value of x. You can use your calculator to determine an approximate value:

$$x = \frac{\log 23}{\log 3} = 2.85$$ (Correct to two decimal places)

Exercise 7.3

Part B

1. Solve the following:

   a. $\log_2 x = 2 \log_2 4$ b. $\log_3 x = 4 \log_3 3$

   c. $2 \log_5 x = \log_5 36$ d. $2 \log x = 4 \log 7$

Knowledge/Understanding

2. Solve the following. Give the answer correct to two decimal places.

   a. $3^x = 5$ b. $5^x = 6$ c. $2^x - 1 = 4$ d. $7 = 12 - 4^x$

3. Solve for x.

   a. $\log x = 2 \log 3 + 3 \log 2$ b. $\log x + \log 3 = \log 1 + \log 4$

   c. $\log x^2 = 3 \log 4 - 2 \log 2$ d. $\log \sqrt{x} = \log 1 - 2 \log 3$

   e. $\log x^{\frac{1}{2}} - \log x^{\frac{1}{3}} = \log 2$ f. $\log_4(x + 2) + \log_4(x - 3) = \log_4 9$

Knowledge/Understanding

4. Solve the following:

   a. $\log_6(x + 1) + \log_6(x + 2) = 1$

   b. $\log_7(x + 2) + \log_7(x - 4) = 1$

c. $\log_2(x + 2) = 3 - \log_2 x$

d. $\log_4 x + \log_4(x + 6) = 2$

e. $\log_5(2x + 2) - \log_5(x - 1) = \log_5(x + 1)$

Communication

5. Explain why there are no solutions to the equations $\log_5(-125) = x$ and $\log_{-2} 16 = x$.

Application

6. A car depreciates at 15% per year. How long is it until it is worth half its original value?

7. Carbon taken from an old animal skeleton contains $\frac{3}{4}$ as much radioactive carbon^{14} (C^{14}) as carbon taken from a present-day bone. How old is the animal skeleton? (The half-life of carbon^{14} is 5760 years.)

Application

8. An isotope of cobalt, Co^{60}, is used in medical therapy. When the radioisotope activity has decreased to 45% of its initial level, the exposure times required are too long and the hospital needs to replace the cobalt. How often does the cobalt need to be replaced? (The half-life of Co^{60} is 5.24 years.)

9. A man wants to sell an old piece of wood to a museum. He claims it came from the stable in which Christ was born 2000 years ago. The museum tests the wood and finds that it contains 4.2×10^{10} atoms of C^{14} per gram. Carbon from present-day wood contains 5.0×10^{10} atoms of C^{14} per gram. Determine the approximate age of the wood. Do you think the relic is authentic? (The half-life of C^{14} is 5760 years.)

Part C

Thinking/Inquiry/ Problem Solving

10. If $\log_2(\log_3 a) = 2$, determine the value of a.

Thinking/Inquiry/ Problem Solving

11. If $\log_{2n}(1944) = \log_n(486\sqrt{2})$, determine the value of n^6.

Section 7.4 — Where We Use Logarithms

LOGARITHMS AND EARTHQUAKES

Earthquakes occur along a fault line, a line where two of the tectonic plates forming the earth's crust meet. Stress builds up between these plates. Eventually the plates slip, resulting in a violent shaking at the earth's surface.

Earthquakes can strike wherever a fault line is located, but the most severe earthquakes occur around the Pacific Rim. Areas most prone to major earthquakes are Japan, Alaska, Taiwan, Mexico, and the western coasts of the United States and Canada.

In 1935, seismologist Charles F. Richter developed a scale to compare the intensities of earthquakes. The amount of energy released in an earthquake is very large, so to avoid using large numbers, a logarithmic scale is used to compare intensities.

The formula Richter used to define the magnitude of an earthquake is

$$M = \log\left(\frac{I}{I_0}\right),$$

where I is the intensity of the earthquake being measured,
I_0 is the intensity of a reference earthquake, and
M is the Richter number used to measure the intensity of earthquakes.

On the Richter scale, the energy of the earthquake increases by powers of 10 in relation to the Richter magnitude number. Earthquakes below magnitude 4 usually cause no damage, and quakes below 2 cannot be felt. A magnitude 6 earthquake is strong, while one of magnitude 7 or higher causes major damage. Below is a list of the five deadliest earthquakes of the twentieth century.

Location	Date	Magnitude	Death Toll
Tangshan, China	July 28, 1976	7.8 to 8.2	240 000
Tokyo, Japan	Sept. 1, 1923	8.3	200 000
Gansu, China	Dec. 16, 1920	8.6	100 000
Northern Peru	May 31, 1970	7.7	70 000
Northern Iran	June 21, 1990	7.3 to 7.7	50 000

EXAMPLE 1

An earthquake of magnitude 7.5 on the Richter scale struck Guatemala on February 4, 1976, killing 23 000 people. On October 2, 1993, an earthquake of magnitude 6.4 killed 20 000 in Maharashtra, India. Compare the intensities of the two earthquakes.

Solution

Let the intensity of the Guatemalan earthquake be I_G and the intensity of the Indian earthquake be I_I. We can use our formula to compare the intensity of the Guatemalan earthquake to the intensity of a reference earthquake (I_0).

$$7.5 = \log\left(\frac{I_G}{I_0}\right)$$

First we solve the expression for I_G:

$$\frac{I_G}{I_0} = 10^{7.5}$$
$$I_G = 10^{7.5} I_0.$$

We use the formula to compare the intensity of the Indian earthquake to the intensity of the reference earthquake (I_0).

$$6.4 = \log\left(\frac{I_I}{I_0}\right)$$

Then solve this expression for I_I:

$$\frac{I_I}{I_0} = 10^{6.4}$$
$$I_I = 10^{6.4} I_0.$$

Now we can compare the intensity of the Guatemalan earthquake to the intensity of the Indian earthquake.

$$\begin{aligned}\frac{I_G}{I_I} &= \frac{10^{7.5} I_0}{10^{6.4} I_0} \\ &= 10^{1.1} \\ &= 12.6 \\ I_G &= 12.6 I_I\end{aligned}$$

The intensity of the Guatemalan earthquake was 12.6 times the intensity of the Indian earthquake.

LOGARITHMS AND SOUND

Our ear is divided into three connecting sections: the outer, middle, and inner ear. The outer ear funnels noise to the eardrum. In the middle ear, three tiny bones

transmit sound to the inner ear. In the inner ear, sound waves are converted to readable nerve impulses by approximately 16 000 hair-like receptor cells, which sway with the sound waves. These cells can be severely damaged by loud sounds, resulting in permanent hearing loss. If you lose one third of these cells, your hearing will be significantly impaired. Hearing loss is progressive. Some hearing loss is inevitable with age, but we would lose much less if we protected our ears at the appropriate times.

The loudness of any sound is measured relative to the loudness of sound at the threshold of hearing. Sounds at this level are the softest that can still be heard.

The formula used to compare sounds is
$L = 10 \log(\frac{I}{I_0})$,
where I is the intensity of the sound being measured,
I_0 is the intensity of a sound at the threshold of hearing, and
L is the loudness measured in decibels $\left(\frac{1}{10}\text{ of a bel}\right)$.

At the threshold of hearing, the loudness of sound is zero decibels (0 dB).

EXAMPLE 2

A sound is 1000 times more intense than a sound you can just hear. What is the measure of its loudness in decibels?

Solution

The loudness of a sound is calculated using the formula $L = 10 \log\left(\frac{I}{I_0}\right)$.

L is the loudness of the sound.
I_0 is the intensity of a sound you can just hear.
I is the intensity of the sound being measured.
$I = 1000\ I_0$

Substituting into the formula:

$$\begin{aligned} L &= 10 \log\left(\frac{1000\ I_0}{I_0}\right) \\ &= 10 \log 1000 \\ &= 10 \times 3 \\ &= 30. \end{aligned}$$

The loudness of the sound is 30 dB.

This table shows the loudness of a selection of sounds.

30 dB	Soft whisper
60 dB	Normal conversation
80 dB	Shouting
90 dB	Subway
100 dB	Screaming child
120 dB	Rock concert
140 dB	Jet engine
180 dB	Space-shuttle launch

Exposure to sound levels of 85 dB during a 35 h work week will eventually cause damage to most ears. The 120 dB volume of the average rock concert will cause the same damage in less than half an hour. The higher the level, the less time it takes before sound-receptor cells start dying and permanent hearing damage occurs. At sound levels of 130 dB, after 75 s you are at risk of suffering permanent damage to your hearing.

EXAMPLE 3 How many more times intense is the sound of normal conversation (60 dB) than the sound of a whisper (30 dB)?

Solution

Let the intensity of the normal conversation be I_n and the intensity of the whisper be I_w. We use our formula to compare the intensity of the normal conversation to the intensity of a sound at the threshold of hearing (I_0).

$$60 = 10 \log\left(\frac{I_n}{I_0}\right)$$

Now solve the expression for I_n.

$$\begin{aligned} 6 &= \log\left(\frac{I_n}{I_0}\right) \\ \frac{I_n}{I_0} &= 10^6 \\ I_n &= 10^6 I_0 \end{aligned}$$

Now we use our formula to compare the intensity of the whisper to the intensity of a sound at the threshold of hearing (I_0).

$$30 = 10 \log\left(\frac{I_w}{I_0}\right)$$

Solve the expression for I_w.

$$\begin{aligned} 3 &= \log\left(\frac{I_w}{I_0}\right) \\ \frac{I_w}{I_0} &= 10^3 \\ I_w &= 10^3 I_0 \end{aligned}$$

Now we can compare the intensity of normal conversation to the intensity of a whisper.

$$\frac{I_n}{I_w} = \frac{10^6 I_0}{10^3 I_0}$$
$$= 10^3$$
$$I_n = 1000 I_w$$

The intensity of normal conversation is 1000 times the intensity of a whisper.

LOGARITHMS AND CHEMISTRY

Chemists measure the acidity of a liquid by determining the concentration of the hydrogen ion $[H^+]$ in the liquid. This concentration is measured in moles per litre. Since this is usually a very small number, a far more convenient measure uses logarithms and is called the pH of a liquid.

Chemists define the acidity of a liquid on a pH scale,
$pH = -\log[H^+]$,
where $[H^+]$ is the concentration of the hydrogen ion in moles per litre.

For distilled water, $[H^+] = 10^{-7}$ mol/L.
To find the pH of distilled water, we proceed as follows:

$$\begin{aligned} pH &= -\log[H^+] \\ &= -\log(10^{-7}) \\ &= -(-7) \\ &= 7. \end{aligned}$$

A liquid with a pH lower than 7 is called an *acid*. A substance with a pH greater than 7 is called a *base*. Chemists calculate the pH of a substance to an accuracy of two decimal places.

EXAMPLE 4

Find the pH of a swimming pool with a hydrogen ion concentration of 6.1×10^{-8} mol/L.

Solution

$$\begin{aligned} pH &= -\log[H^+] \\ &= -\log(6.1 \times 10^{-8}) \\ &= 7.21 \text{ (correct to two decimal places)} \end{aligned}$$

(pH is given to two decimal places)

Alternate Solution

$$\begin{aligned} pH &= -\log(6.1 \times 10^{-8}) \\ &= -(\log 6.1 + \log 10^{-8}) \\ &= -(.79 - 8) \\ &= 7.21 \end{aligned}$$

The pH of the pool is 7.21.

EXAMPLE 5 The pH of a fruit juice is 3.10. What is the hydrogen ion concentration of the fruit juice?

Solution

$$\begin{aligned} pH &= -\log[H^+] \\ 3.10 &= -\log[H^+] \\ \log[H^+] &= -3.10 \\ [H^+] &= 10^{-3.10} \\ &= 0.000\ 79 \end{aligned}$$

The hydrogen ion concentration is 7.9×10^{-4} mol/L.

Exercise 7.4

Part B

Communication

1. It is interesting to note the inclusion of the negative sign in the formula for pH. Discuss reasons why this makes sense.

Knowledge/Understanding

2. If one earthquake has a magnitude of 5 on the Richter scale and a second earthquake has a magnitude of 6, compare the intensities of the two earthquakes.

3. A sound is 1 000 000 times more intense than a sound you can just hear. What is the loudness of the sound?

Knowledge/Understanding

4. Find the pH of a liquid with a hydrogen ion concentration of 8.7×10^{-6} mol/L.

Application

5. An earthquake of magnitude 2 cannot be felt. An earthquake of magnitude 4 will be noticed but usually causes no damage. Compare the intensities of two such earthquakes.

Application

6. An earthquake in Gansu, China, on December 16, 1920, measured 8.6 on the Richter scale and killed 100 000 people. An earthquake that usually causes no damage measures 4 on the Richter scale. Compare the intensities of the two earthquakes.

Communication

7. An earthquake in the Quetta area of Pakistan on May 31, 1935, measured 6.8 on the Richter scale. This quake killed 50 000 people. On October 2, 1987, an earthquake of magnitude 6.1 shook Los Angeles, California, and killed six people.

a. Compare the magnitude of the two earthquakes.

b. Why do you think the death toll was so much higher with the earthquake in Pakistan?

8. On January 24, 1939, an earthquake measuring 8.3 occurred in Chillan, Chile, killing 28 000 people. On September 21, 1999, an earthquake in Taiwan measured 7.6 on the Richter scale and killed 2100 people. Compare the intensities of these two earthquakes.

Thinking/Inquiry/ Problem Solving

9. Sasha needs a new muffler on her car. She has been told that the sound from her car was measured at 120 dB. After installing the new muffler, the loudness of her car is 75 dB. How many times more intense was the sound from her defective muffler?

Thinking/Inquiry/ Problem Solving

10. Tania's infant daughter has colic and cries during the night. The noise level in the house at these times is 75 dB. When the baby finally falls asleep, the noise level is 35 dB. How many times more intense is the noise level in the house when the baby is crying?

11. How many times more intense is the sound of a space-shuttle launch (180 dB) than the sound of a jet engine (140 dB)?

Thinking/Inquiry/ Problem Solving

12. Jonathan lives near a busy street. He has all the windows in his home open and measures the noise level inside as 79 dB. He closes the windows and finds the noise level is 68 dB. By what factor did the intensity of the noise decrease when Jonathan closed the windows?

Application

13. Find the hydrogen ion concentration of milk, which has a pH of 6.50.

14. Find the hydrogen ion concentration of milk of magnesia, which has a pH of 10.50.

Section 7.5 — Change of Base

We have avoided two types of questions so far, and there is a very good reason for doing so. We can determine $\log_2 64$ because $2^6 = 64$; however, we cannot easily express 63 as a power of 2, which makes determining $\log_2 63$ more challenging. We cannot use a calculator because the logarithm operation only determines logarithms to base 10.

For a similar reason, we cannot easily use a calculator to obtain the graph of $y = \log_5 x$.

What is to be done? It turns out that the solution lies in our ability to use the properties of logarithms so that we can always use base 10.

EXAMPLE 1 Determine $\log_2 63$.

technology

Solution

Let $\log_2 63 = y$.
Then $2^y = 63$.

Taking logarithms of both sides,

$$\log 2^y = \log 63 \text{ (using base 10)}$$

$$y \log 2 = \log 63$$

$$y = \frac{\log 63}{\log 2}.$$

Then $\log_2 63 = \dfrac{\log 63}{\log 2} = 5.977$ (correct to three decimals).

Can this be done in any situation? Let's consider a general case and prove that $\log_b x = \dfrac{\log_a x}{\log_a b}$, where $a > 0$.

Proof

Let $\log_b x = y$.
From the definition, $b^y = x$.
Taking logarithms of both sides, using base a,

$$\log_a b^y = \log_a x$$

$$y \log_a b = \log_a x$$

$$y = \frac{\log_a x}{\log_a b}.$$

Then $\log_b x = \dfrac{\log_a x}{\log_a b}$.

Since this is true for any base a, it is certainly true for the particular base 10, and we can use this to determine logarithms given any base.

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

EXAMPLE 2 Use your calculator to find the value of $\log_3 23$, correct to two decimal places.

Solution

$$\begin{aligned}\log_3 23 &= \frac{\log 23}{\log 3}\\ &= 2.8540\end{aligned}$$

Correct to two decimal places, $\log_3 23 = 2.85$.

EXAMPLE 3 Prove that $\log_t b = \dfrac{1}{\log_b t}$.

Proof

$$\begin{aligned}\log_t b &= \frac{\log b}{\log t}\\ &= \frac{1}{\frac{\log t}{\log b}}\\ &= \frac{1}{\log_b t}\end{aligned}$$

$$\log_t b = \frac{1}{\log_b t}$$

EXAMPLE 4 Show that $\dfrac{1}{\log_3 a} + \dfrac{1}{\log_4 a} = \dfrac{1}{\log_{12} a}$.

Solution

$$\begin{aligned}\frac{1}{\log_3 a} + \frac{1}{\log_4 a} &= \log_a 3 + \log_a 4\\ &= \log_a 12\\ &= \frac{1}{\log_{12} a}\end{aligned}$$

Therefore, $\dfrac{1}{\log_3 a} + \dfrac{1}{\log_4 a} = \dfrac{1}{\log_{12} a}$.

EXAMPLE 5 If $a^2 + b^2 = 14ab$, where $a > 0$, $b > 0$, show that $\log\left(\frac{a+b}{4}\right) = \frac{1}{2}(\log a + \log b)$.

Solution

Since
$$a^2 + b^2 = 14ab$$
$$a^2 + 2ab + b^2 = 16ab$$
$$(a+b)^2 = 16ab$$
$$\left(\frac{a+b}{4}\right)^2 = ab.$$

Taking logarithms of both sides,

$$2\log\left(\frac{a+b}{4}\right) = \log(ab)$$
$$= \log a + \log b.$$

Then $\log\left(\frac{a+b}{4}\right) = \frac{1}{2}(\log a + \log b)$.

We noted at the beginning of this discussion that the graph of $y = \log_b x$ is not immediately accessible by calculator. Using the change of base formula, it is easy to use a graphing calculator to obtain such a graph.

EXAMPLE 6 Use a graphing calculator to graph each of the following:

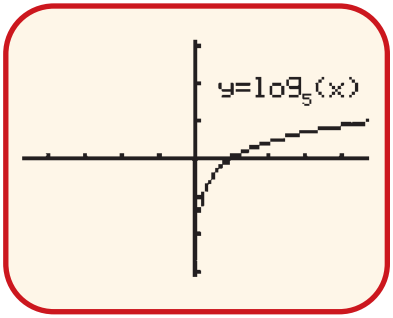
a. $y = \log_5 x$ b. $y = \log_{0.5} x$

Solution

technology

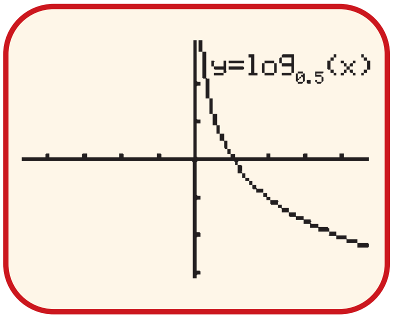
a. $\log_5 x = \frac{\log x}{\log 5} = \frac{1}{\log 5}\log x$

Input $y = \frac{1}{\log 5}\log x$ and graph the function.



b. $\log_{0.5} x = \frac{\log x}{\log 0.5} = \frac{1}{\log 0.5}\log x$

Input $y = \frac{1}{\log 0.5}\log x$ and graph the function.



Exercise 7.5

Part B

Knowledge/Understanding

1. Use your calculator to find the value of each of the following, correct to three decimal places.

a. $\log_5 21$

b. $\log_7 124$

c. $\log_6 3.24$

d. $\log_4 4.7$

Application

2. Show that each of the following statements is true.

a. $\frac{1}{\log_5 a} + \frac{1}{\log_3 a} = \frac{1}{\log_{15} a}$

b. $\frac{1}{\log_8 a} - \frac{1}{\log_2 a} = \frac{1}{\log_4 a}$

c. $\frac{2}{\log_6 a} = \frac{1}{\log_{36} a}$

d. $\frac{2}{\log_8 a} - \frac{4}{\log_2 a} = \frac{1}{\log_4 a}$

Application

3. Sketch the graph of each of the following:

a. $y = \log_3 x$

b. $y = 4\log_2 x$

c. $y = \log_{0.5} x$

d. $y = \log_{0.2} x^2$

Communication

4. Describe the changes to the graph of $y = \log_3 x$ when x is replaced by x^2.

Thinking/Inquiry/Problem Solving

5. For $a > 1$, $b > 1$, show that $(\log_a b)(\log_b a) = 1$.

Part C

6. If $a^2 + b^2 = 23ab$, where $a > 0$, $b > 0$, show that $\log\left(\frac{a+b}{5}\right) = \frac{1}{2}(\log a + \log b)$.

Thinking/Inquiry/Problem Solving

7. For $a > 0$, $a \neq 1$, $x > 0$, prove that $\log_a \frac{1}{x} = \log_{\frac{1}{a}} x$.

8. If $\log_a b = p^3$ and $\log_b a = \frac{4}{p^2}$, show that $p = \frac{1}{4}$.

9. If $a^3 - b^3 = 3a^2b + 5ab^2$, where $a > 0$, $b > 0$, show that $\log\left(\frac{a-b}{2}\right) = \frac{1}{3}(\log a + 2\log b)$.

Key Concepts Review

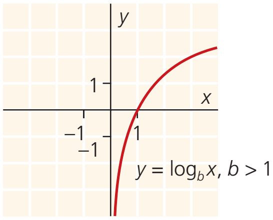
In Chapter 7, you have learned that the logarithmic function is the inverse of the exponential function, and that the logarithmic function is usually written as $y = \log_b x$. You should now know how to solve logarithmic equations as well as where we use logarithms. You should also be familiar with Change of Base formulas. Here is a brief summary of key chapter concepts.

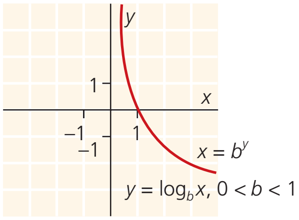
The Logarithmic Function

Exponential Form		**Logarithmic Form**	
$x = b^y$	$\longleftrightarrow$	$y = \log_b x$	$b > 0$ and $b \neq 1$

The logarithm of a number x with a given base is the exponent to which that base must be raised to yield x.

As with the exponential function, there are two possible versions of the graph of the logarithmic function:





Basic Properties of Logarithms

- $\log_b 1 = 0$
- $\log_b b = 1$
- $\log_b b^x = x$
- $b^{\log_b x} = x$

Properties of Logarithms

For $x > 0$, $w > 0$, and r a real number:

- $\log_a xw = \log_a x + \log_a w$
- $\log_a\left(\frac{x}{w}\right) = \log_a x - \log_a w$
- $\log_a x^r = r \log_a x$

Change of Base Formulas

- $\log_b x = \dfrac{\log_a x}{\log_a b}$
- $\log_b x = \dfrac{1}{\log_x b}$

Career Link wrap-up — investigate and apply

CHAPTER 7: MEASURING ON A LOGARITHMIC SCALE

Many Northern Ontario communities are familiar with the devastation acid rain can bring to lake ecosystems. While the major contributors to acid rain are oxides of sulfur and nitrogen generated in industrial processes, carbon dioxide, the so-called "greenhouse gas" that is thought by many to cause global warming, also can contribute to acid rain. Carbon dioxide can acidify water in the atmosphere just as it acidifies the water used to make our favourite soft drinks when it is bubbled through liquids to make carbonated beverages.

Acidity is measured on the pH scale, which is defined as

$$pH = -\log[H^+]$$

where $[H^+]$ is the concentration of the hydrogen ion in moles per litre. The hydrogen ion, which causes acidity, is a function of the percentage of carbon dioxide in the atmosphere, an amount that can be elevated through the combustion of fossil fuels, shown by

$$\frac{[H^+]^2}{(CO_2)} = K,$$

where CO_2 is the percentage of carbon dioxide in the atmosphere and K is a constant (1.52×10^{-10}).

a. Use log rules to show that $pH = -\frac{1}{2}[\log(CO_2) + \log(K)]$ and determine the change in pH if the percentage of CO_2 increases from 0.03% to 0.06%. Is this a significant change? Explain.

b. Aquatic toxicologists conduct research on the response of fish and other aquatic species to the environmental contamination of their ecosystem. The data in the table was collected in a lab study to examine the effect of pH on a fish population.

Obtain an algebraic expression for population as a function of pH by linearizing the population data in the table and obtaining an equation of the form $P = 10^{k(pH)+d}$, where k is the growth rate factor and d is a constant. Verify your equation by looking at doubling patterns in the table and plotting both equations on the graphing calculator.

pH	Population
4.0	250
4.5	353
5.0	500
5.5	707
6.0	1000

c. An environmental assessment has established that a 25% decline in a population due to decreased pH is tolerable. Use your model from part **b** to determine how low the pH can drop. What percentage of carbon dioxide does this correlate to? The initial pH is 6.0 and the initial population is 1000.

Review Exercise

1. Evaluate each of the following:

   a. $\log_3 27$ b. $\log_5 \frac{1}{125}$ c. $\log_4 32$ d. $\log_6 \sqrt[3]{36}$

2. Evaluate each of the following:

   a. $\log_6 9 + \log_6 4$ b. $\log_2 3.2 + \log_2 100 - \log_2 5$

   c. $\log_5 \sqrt[3]{25} - \log_3 \sqrt[3]{27}$ d. $7^{\log_7 5}$

3. Solve each of the following equations:

   a. $3 = \log_2\left(\frac{1}{y}\right)$ b. $\log(x + 3) + \log x = 1$

   c. $\log_5(x + 2) - \log_5(x - 1) = 2 \log_5 3$ d. $\frac{\log(35 - x^3)}{\log(5 - x)} = 3$

4. Compare the intensities of an earthquake of magnitude 7.2 on the Richter scale that occurred in Kobe, Japan, on January 17, 1995, to an earthquake of magnitude 6.9 that occurred in northwest Armenia on December 7, 1988.

5. The noise in the school cafeteria is recorded at 50 dB at 10:00. At 12:00 the noise is found to be 100 dB. By what factor does the intensity of the sound increase at lunchtime?

6. A liquid has a pH of 5.62. Find the hydrogen ion concentration $[H^+]$.

7. Describe the transformation that takes the graph of $y = \log_4 x$ to the graph of $y = \log_4(16x^2)$.

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8. Use your calculator to find the value of each of the following, correct to three decimal places.

   a. $\log_{19} 264$ b. $\log_5 34.62$

9. Show that $\frac{2}{\log_9 a} - \frac{1}{\log_3 a} = \frac{3}{\log_3 a}$.

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10. Use a graphing calculator to graph each of the following:

    a. $y = \log_7 x$ b. $y = 2 \log_6(6x)$

Chapter 7 Test

Achievement Category	Questions
Knowledge/Understanding	All questions
Thinking/Inquiry/Problem Solving	8, 10, 11
Communication	3, 5
Application	2, 4, 6, 7, 9

1. Evaluate each of the following:
   a. $\log_3 27$
   b. $\log_5 125$
   c. $\log_2 \frac{1}{16}$
   d. $\log_5 \sqrt[4]{25}$
   e. $\log_2 8 + \log_3 9$
   f. $\log_3 9^{\frac{1}{3}}$
2. Evaluate each of the following:
   a. $\log_2 \frac{8}{5} + \log_2 10$
   b. $\log_6 108 - \log_6 3$
3. Describe the effect on the graph of $y = \log_5 x$ when x is replaced by $25x^2$.
4. Solve the following equations:
   a. $2 \log x = 3 \log 4$
   b. $\log x + \log 3 = \log 12$
   c. $\log_2(x + 2) + \log_2 x = 3$
   d. $\log_2(x - 2) + \log_2(x + 1) = 2$
5. Explain why there are no solutions to the equation $\log_3(-9) = x$.
6. A radioactive substance decays from 20 g to 15 g in 7 h. Determine the half-life of the substance.
7. An earthquake of magnitude 8.3 on the Richter scale killed 200 000 in Tokyo, Japan, on September 1, 1923. On February 4, 1976, an earthquake of magnitude 7.5 killed 23 000 in Guatemala. Compare the intensity of these earthquakes.
8. Kendra is talking to her friend on the subway platform, where the noise level is 60 dB. When a subway train enters the station, Kendra can no longer hear her friend. The noise level from the train is 90 dB. How many times more intense is the noise level in the station when the train enters?
9. A liquid has a pH of 8.31. Find the hydrogen ion concentration $[H^+]$.
10. Show that the following statement is true: $\frac{3}{\log_2 a} = \frac{1}{\log_8 a}$.
11. If $\log_a b = \frac{1}{x}$ and $\log_b \sqrt{a} = 3x^2$, show that $x = \frac{1}{6}$.

Cumulative Review

CHAPTERS 5–7

1. Find $\frac{dy}{dx}$ for the following:
   a. $x^2 + y^2 = 324$
   b. $4x^2 - 16y^2 = 64$
   c. $x^2 + 16y^2 = 5x + 4y$
   d. $2x^2 - xy + 2y = 5$
   e. $\frac{1}{x} + \frac{1}{y} = 1$
   f. $(2x + 3y)^2 = 10$

2. Find an equation of the tangent to the curve at the indicated point.
   a. $x^2 + y^2 = 13$ at $(-2, 3)$
   b. $x^3 + y^3 = y + 21$ at $(3, -2)$
   c. $xy^2 + x^2y = 2$ at $(1, 1)$
   d. $y^2 = \frac{3x^2 + 9}{7x^2 - 4}$ at $(1, 2)$

3. Find f' and f'' for the following:
   a. $f(x) = x^5 - 5x^3 + x + 12$
   b. $f(x) = \frac{-2}{x^2}$
   c. $f(x) = \frac{4}{\sqrt{x}}$
   d. $f(x) = x^4 - \frac{1}{x^4}$

4. Find $\frac{d^2y}{dx^2}$ for the following:
   a. $y = x^5 - 5x^4 + 7x^3 - 3x^2 + 17$
   b. $y = (x^2 + 4)(1 - 3x^3)$

5. The displacement at time t of an object moving along a line is given by $s(t) = 3t^3 - 40.5t^2 + 162t$ for $0 \leq t \leq 8$.
   a. Find the position, velocity, and acceleration.
   b. When is the object stationary? advancing? retreating?
   c. At what time t is the velocity not changing?
   d. At what time t is the velocity decreasing; that is, the object is decelerating?
   e. At what time t is the velocity increasing; that is, the object is accelerating?

6. A particle moving on the x-axis has displacement $x(t) = 2t^3 + 3t^2 - 36t + 40$.
   a. Find the velocity of the particle at time t.
   b. Find the acceleration of the particle at time t.
   c. Determine the total distance travelled by the particle during the first three seconds.

7. For each of the following cost functions, in dollars, find
   a. the cost of producing 900 items.
   b. the average cost of each of the first 900 items produced.
   c. the marginal cost when $x = 900$, and the cost of producing the 901st item.
      i) $C(x) = 5x + 100$ ii) $C(x) = \sqrt{x} + 8000$

8. The total cost of producing x units of a certain commodity is given by the function $C(x) = 3x^2 + x + 48$.
   a. Determine the average cost of 3, 4, 5, and 6 units of the commodity.
   b. Using a graphing utility, show that the minimum average cost is \$25 when 4 units are produced.

9. Find the indicated rate for each of the following:
   a. Find $\frac{dy}{dt}$, where $x^2 + y^2 = 36$ and $\frac{dx}{dt} = 4$, when $x = 3$.
   b. Find $\frac{dy}{dt}$, where $5x^2 - y = 100$ and $\frac{dx}{dt} = 10$, when $x = 10$.

10. An environmental study of a suburban community suggests that t years from now, the average level of carbon monoxide in the air will be $q(t) = 0.05t^2 + 0.1t + 3.4$ parts per million.
    a. At what rate will the carbon monoxide level be changing with respect to time one year from now?
    b. By how much will the carbon monoxide level change in the first year?

11. Suppose a spherical piece of ice is melting at a rate of 5 cm^3/min and retains its spherical shape at all times. How fast is the radius changing at the instant when the radius is 4 cm? How fast is the surface area of the sphere changing at the same instant?

12. Sand is being dumped on a pile in such a way that it always forms a cone whose radius equals its height. If the sand is being dumped at a rate of 10 m^3/h, at what rate is the height of the pile increasing when there are 1050 m^3 of sand in the pile?

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13. Graph each of the following:
    a. $y = 2^x$ b. $y = 10^x + 1$ c. $y = 5^{x-2} + 3$
    d. $y = \left(\frac{1}{2}\right)^{x+3}$ e. $y = 3e^{x-1} - 2$ f. $y = 8 - e^{-x}$

14. Simplify each of the following:

a. $\dfrac{(27)^{\frac{1}{3}} \bullet 4^2}{48}$ b. $4 \bullet 2^{x-1}$ c. $e^{\sqrt{25}} \bullet e^{-5}$

d. $9\left(\dfrac{1}{3}\right)^{-3}$ e. $\dfrac{e^3 e^{-2x}}{e^{-x}}$ f. $(e^{4x})^3$

15. Solve each equation, if possible.

a. $5^{2x+9} = 125$ b. $3^{x^2+3} = 81^x$

c. $\left(\dfrac{1}{4}\right)^{x+3} = \left(\dfrac{1}{8}\right)^{x-1}$ d. $2^{2x} - 12(2^x) + 32 = 0$

e. $e^x = 1$ f. $e^{2x} + e^x - 2 = 0$

16. Digital cable is being introduced into a certain city. The number of subscribers t months from now is expected to be $N(t) = \dfrac{80\,000}{1 + 10e^{-0.2t}}$.

a. How many subscribers will there be after six months?

b. How many subscribers will there eventually be?

17. A rumour spreads through a school. After the rumour has begun, $N(t) = \dfrac{50}{1 + 49e^{-t}}$ people have heard the rumour where t is in hours. How many people have heard it after 4 h?

18. Assume that the annual rate of inflation will average 5% over the next ten years.

a. Write an equation to represent the approximate cost, C, of goods or services during any year in that decade.

b. If the price of a mechanical inspection for your car is presently \$39.95, estimate the price ten years from now.

c. If the price of an oil change ten years from now is \$40.64, determine the price of an oil change today.

19. The value of a new car depreciates at a rate of 25% per year.

a. Write an equation to represent the approximate value, V, of a car purchased for \$30 000.

b. Determine the value of the car two years after it is purchased.

c. Approximately how many years will it take until the car is worth \$3000?

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20. Find an exponential function modelled by the experimental data collected over time t.

t	0	1	2	3	4
y	1200	720	432	259.2	155.52

21. Graph each of the following:

a. $y = \log_2 x$ b. $y = \log x$ c. $y = 3\log(2 - x)$

22. Find the value of each of the following:

a. $\log 100$ b. $\log_2 16$ c. $\log_3 243$ d. $\log 0.001$

e. $\log_8 2$ f. $\log_4\left(\frac{1}{8}\right)$ g. $\log 10$ h. $\log 2.2$

i. $\log_a \frac{1}{a^2}$ j. $4^{\log_4 7}$ k. $10^{-10\log 3}$ l. $a^{8\log_a \sqrt{a}}$

23. Use the properties of logarithms to write each expression as a sum, difference, and/or multiple of logarithms.

a. $\log\frac{2}{3}$ b. $\log\frac{xy}{z}$ c. $\log\frac{1}{5}$

d. $\log\sqrt{\frac{x+1}{x-1}}$ e. $\log\left(\frac{x^2-4}{x^5}\right)^4$ f. $\log_a 4a^5$

24. Write the expression as a logarithm of a single quantity.

a. $\log(x - 4) + \log(3 - x)$ b. $3\log_2 x + 2\log_2 y - 4\log_2 z$

c. $2\log 3 - \frac{1}{2}\log(x^2 + 1)$ d. $\log x - 4\log(x - 5) + \frac{2}{3}\log\sqrt{x + 1}$

25. Use the formula $\log_a x = \frac{\log x}{\log a}$ to determine the value of each of the following logarithms:

a. $\log_2 12$ b. $\log_3 \frac{1}{2}$ c. $\log_3 8$

d. $\log_8 4$ e. $\log_4 6$ f. $\log_{\frac{1}{2}} 15$

26. Solve each of the following equations:

a. $x = \log_5 125$ b. $x = \log_5 225 - \log_5 9$

c. $x - 3\log_3 243 = 4\log_2\sqrt{512}$ d. $\log_5(2x + 5) = 2$

e. $2\log_3(4x + 1) = 4$ f. $\log_{12} x - \log_{12}(x - 2) + 1 = 2$

g. $2^x = 7$ h. $\log 10^x = -1$

i. $\log(x - 4) = 1$ j. $(\log x)^2 + 3\log x - 10 = 0$

27. The level of sound in decibels is $SL = 10\log(I \times 10^{12})$ where I is the intensity of the sound in watts per square metre (W/m^2). A decibel, or dB, named for Alexander Graham Bell, is the smallest increase of the loudness of a sound that is detectable by the human ear.

a. What is the sound level when the intensity is 2.51×10^{-5} W/m^2?

b. The threshold of pain is 120 dB. A room with appliances on has an intensity of 6.31×10^{-4}. Is the sound level in the room bearable to the human ear?

c. Write the intensity of sound of normal conversation, 50 dB, in scientific notation.

d. Calculate the intensity of the sound at a rock concert where the sound level is 110 dB.